UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH1101**

ASSESSMENT PATTERN : **MATH1101B**

MODULE NAME : **Analysis 1**

DATE : **20-May-11**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2010/11-MATH1101B-001-EXAM-204

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State what it means for a real sequence to converge.

   (b) Use the definition of convergence (not the combination theorem or any other theorems) to prove that

$$\lim_{n\to\infty} \frac{n^2+1}{2+3n^2} = \frac{1}{3}$$

   (c) State what it means for a real sequence to be a Cauchy sequence.

   (d) Use the definition (not a theorem) to show that the sequence $\langle a_n \rangle$ given by

$$a_n = \frac{1}{n}$$

   is a Cauchy sequence.

   (e) Prove that every convergent sequence of reals is a Cauchy sequence.

2. (a) State the definition of $\lim_{x\to a^+} f(x) = l$.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 - 3, & (x < 2), \\ 6/x, & (x \geq 2). \end{cases}$$

   Show carefully (using ϵ and δ) that

$$\lim_{x\to 2^-} f(x) = 1, \quad \lim_{x\to 2^+} f(x) = 3.$$

   (c) Let f be continuous on the compact interval $[a,b]$. Show that f is bounded on $[a,b]$.

   (d) Can you apply the theorem in (c) to the function f in (b) on the interval $[0,5]$? Determine (with explanation) whether the function f is bounded on $[0,5]$ or not.

3. (a) State and prove the sandwich theorem for sequences.

(b) Show that $\lim_{n\to\infty} \sqrt[n]{3^n + 5^n} = 5$.

(*Hint*: You may assume that $\lim_{n\to\infty} \sqrt[n]{a} = 1$ for $a > 0$.)

(c) Define what it means for the series $\sum_{n=1}^{\infty} a_n$ to converge.

(d) Determine with explanations whether the following series converge or diverge.

$$\sum_{n=1}^{\infty} n^3 \left(\frac{1}{2}\right)^n, \quad \sum_{n=1}^{\infty} \sqrt[n]{3^n + 5^n}.$$

4. (a) (i) State the Cauchy–Schwarz inequality.

(ii) Let $x_1, x_2, \ldots, x_n$ and $w_1, w_2, \ldots, w_n$ be positive numbers with $\sum_{j=1}^{n} w_j^2 = 1$. Use the Cauchy–Schwarz inequality to show that

$$\left(\sum_{j=1}^{n} x_j \cdot w_j^2\right)^2 \leq \sum_{j=1}^{n} (x_j^2 \cdot w_j^2).$$

(iii) If the series $\sum_{n=1}^{\infty} a_n^2$ converges, show that the series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^{3/2}}$$

converges absolutely.

(b) State and prove the Bolzano–Weierstrass Theorem. You may assume that every sequence of reals has a monotone subsequence.

5. (a) State the Intermediate Value Theorem.

(b) Let $f : [0,1] \to [0,1]$ be continuous on $[0,1]$.
Prove that for some $\xi \in [0,1]$ we have $f(\xi) = \xi$.

(c) Let y be positive. Using the function $f(x) = x^2$, show that y has a square root.

(d) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function for which

$$(f(x))^2 - x^2 = 1, \quad \forall x \in \mathbb{R},$$

and $f(0) = -1$. Show that

$$f(x) = -\sqrt{1 + x^2}, \quad \forall x \in \mathbb{R}.$$

6. (a) (i) Show that for all positive numbers y we have

$$\ln(y) \leq y - 1.$$

You may assume that $e^x \geq x + 1$ for all $x \in \mathbb{R}$.

(ii) State and prove the Arithmetic Mean – Geometric Mean Inequality for n non-negative numbers $a_1, a_2, \ldots, a_n$. You may use (i).

(b) (i) For $0 < x < y$ prove the following inequalities

$$x < \frac{2}{\frac{1}{x} + \frac{1}{y}} < \sqrt{xy} < y.$$

(ii) Define the sequence $\langle x_n \rangle$ by

$$x_1 = 1/2, \quad x_2 = 1, \quad x_{2n+1} = \sqrt{x_{2n} x_{2n-1}}, \quad x_{2n+2} = \frac{2}{\frac{1}{x_{2n}} + \frac{1}{x_{2n+1}}}, \quad n \geq 1.$$

Use induction and (i) to prove that

$$x_{2n-1} < x_{2n+1} < x_{2n+2} < x_{2n}, \quad n \in \mathbb{N}.$$

Deduce that the subsequences given by $\langle x_{2n} \rangle$ and $\langle x_{2n-1} \rangle$ converge to the same limit. What does this imply for the sequence $\langle x_n \rangle$?